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Hadron Masses in Quantum Chromodynamics on the Transverse Lattice

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ABSTRACT

Calculational methods are formulated for the transverse lattice version of Quantum Chromodynamics. These methods are used to study the low lying spectrum of gluon bound states in the pure Yang-Mills theory.

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I. INTRODUCTION

It has been proposed that the strong interaction aspects of particle dynamics can be described by a Lagrangian field theory, Quantum Chromodynamics (QCD). The theory consists of colored quarks interacting via colored gluons. The original applications of the theory to short distance and light cone dominated processes have been extended to encompass various inclusive processes[1] ($e^+e^- \to$ n jets, $pp \to \gamma^+\gamma^- X$ and others) and maybe even certain exclusive processes[2] (elastic form factors of hadrons). Renormalization group equations and general factorization properties were used to analyze those phenomena perturbatively.

The applications of the same methods to study the large distance structure of the theory have not resulted in the emergence of a spectrum consisting of only color singlet bound states.[3] Various attempts have been made to incorporate non-perturbative effects in QCD. Semiclassical methods utilizing instantons and other configurations have important consequences for the chiral structure of QCD and may even result in an effective MIT like bag theory.[4] The strong coupling aspects are emphasized by reformulating QCD in terms of a lattice gauge theory. Wilson[5] has proposed a four-dimensional eulcidean lattice version of QCD while Kogut and Susskind[6] have studied a Hamiltonian formulation of QCD. Within those lattice theories bound state spectra have been calculated in a strong coupling expansion. This was done for pure Yang-Mills theory[7] and for colored quarks interacting[5,8] with fermions. These calculations are qualitatively successful for aspects not involving chiral symmetry and seem to result in a string-like picture for excited bound states. Finally 't Hooft[9] has proposed the $1/N_c$ expansion, where the color group is taken to be $SU(N_c)$, which leads to a pictorial simplification of the theory. The task of summing the surviving planar diagrams still seems formidable.

However if QCD confines, then for large N_c a valence quark picture emerges for the meson sector which will consist of an infinite number of stable hadrons. The pure glueball sector will also contain stable hadrons decoupled from mesons.[10] Another approach to QCD was discussed by Bardeen and Pearson[11] (to be referred to as I). It will be reviewed in section II. Its structure is nontrivial in both the weak and strong coupling regimes. In this paper we discuss the calculation of hadronic glueball masses in the strong coupling regime. In section III we discuss the non-perturbative longitudinal dynamics of the model. Bare hadrons are constructed from gauge potentials and "real" color degrees of freedom. In section IV, a perturbative strong coupling analysis of an effective field theory for bare hadrons is performed resulting in hadrons with a transverse motion. The calculations are discussed in section V.

II. TRANSVERSE LATTICE VERSION OF QCD

In this section we will review the transverse lattice version of QCD given in I. First we will present the lattice action functional in terms of the link variables introduced by Wilson.[5] Then we will discuss the transformation to linearized degrees of freedom. Finally we will quantize the resulting linear theory. Since this paper is only concerned with the pure Yang-Mills sector of the theory we will not discuss here any of the problems associated with describing fermions on the lattice.

If we use the matrix form of the gauge field

$$A_\mu \equiv igA_\mu^a T^a \tag{2.1}$$

where the group generators are normalized by the conditions

$$\mathrm{tr}(T^aT^b) = \delta^{ab}/2 \quad , \quad [T^a, T^b] = i f^{abc}T^c \quad , \tag{2.2}$$

then the QCD action has the form

$$A = \int d^4x \frac{1}{2g^2} \mathrm{tr}(G^{\mu\nu}G_{\mu\nu}) \tag{2.3}$$

where the Yang-Mills field strength is given by

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad . \tag{2.4}$$

Since there are only two dynamical gauge fields we choose a way of putting the theory on the lattice, gauge, and quantization procedure which exploits this fact. First we choose to leave the longitudinal coordinates t and z and the longitudinal gauge fields A_t and A_z intact while we make the transverse coordinates discrete. Thus

$$\vec{x}_\perp = a(n_x, n_y) \tag{2.5}$$

where a is the lattice spacing and n_x and n_y are integers. The transverse gauge fields A_x and A_y are replaced by link variables $U_{\vec{x}_\perp,\alpha}(t, z)$ which are associated with the link between the lattice sites $\vec{x}_\perp$ and $\vec{x}_\perp + \hat{\alpha}$ where $\alpha = 1,2$ and $\hat{\alpha} = (a, 0)$ or $(0, a)$. The U's also depend on the longitudinal coordinates t and z. The U's may be interpreted as the phase factors

$$\exp\left[ig \int_{\vec{x}_\perp}^{\vec{x}_\perp + \hat{\alpha}} A_\beta dx^\beta \right] \quad . \tag{2.6}$$

We may now write down a discrete version of the QCD action which reproduces (2.3) in the naive continuum limit ($ga \to 0$) with the expression (2.6) is substituted for the U's and the exponential is expanded out in powers of a, viz.

$$A = \int dzdt \Bigg[\sum_{\vec{x},\mu,\nu} \frac{a^2}{2g^2} \mathrm{tr}(G^{\mu\nu}G_{\mu\nu})$$

$$+ \sum_{\vec{x},\alpha,\mu} \frac{1}{g^2} \mathrm{tr}(D_\mu U_{\vec{x}_\perp,\alpha} D^\mu U^\dagger_{\vec{x}_\perp,\alpha})$$

$$+ \sum_{\vec{x}_\perp,\alpha,\beta} \frac{1}{g^4 a^2} \mathrm{tr}(U_{\vec{x}_\perp,\alpha} U_{\vec{x}_\perp+\hat{\alpha},\beta} U^\dagger_{\vec{x}_\perp+\hat{\beta},\hat{\alpha}} U^\dagger_{\vec{x}_\perp,\beta}) \Bigg] \tag{2.7}$$

where $\mu, \nu = 0,3$ and $\alpha, \beta = 1,2$. The (longitudinal) covariant derivative of the link variables is given by

$$D_\mu U_{\vec{x}_\perp,\alpha} = (\partial_\mu + A_\mu(\vec{x}_\perp))U_{\vec{x}_\perp,\alpha} - U_{\vec{x}_\perp,\alpha} A_\mu(\vec{x}_\perp + \hat{\alpha}) \quad . \tag{2.8}$$

The action for the longitudinal gauge fields is unchanged from the continuum except to replace integrals by sums. The action for the purely transverse gauge fields is the familiar "plaquette" action used before in lattice gauge theories. The mixed term involving $D_\mu U$ is the simplest local interaction which reproduces the corresponding mixed term in the continuum. Most importantly the lattice action (2.7) remains gauge invariant under the restriction of the original gauge group to the lattice. We can use this gauge invariance to completely eliminate the longitudinal gauge fields A_μ from the theory. This may be accomplished by using light-cone gauge $(A_- = (A_0 - A_3)/\sqrt{2} = 0)$ with light-cone quantization $(\tau = x^+ = (x^0 + x^3)/\sqrt{2})$ so that A_+ becomes a parametric field which may be eliminated by its equations of constraint in favor of a nonlocal Coulomb interaction. In this respect the treatment at each transverse lattice site is the same as the discussion of two dimensional QCD given by several authors.[12] Explicitly setting A_- to zero, the terms in (2.7) which depend on A_+ become

$$A = \int dx^+ dx^- \sum_{\vec{x}} \Big[\frac{a^2}{2} (\partial_- \vec{A}_+)^2 + g\vec{A}_+ \cdot \vec{J}_- + \sum_\alpha \frac{1}{g^2} \operatorname{tr} \left(\partial_\mu U_{\vec{x}_\perp,\alpha} \partial^\mu U^\dagger_{\vec{x}_\perp,\alpha} \right) + \cdots \Big] \tag{2.9}$$

where we are using the Hermitian form of $\vec{A}_+$ in the vector representation and the current $\vec{J}_-$ is given by

$$\vec{J}_- = \frac{1}{g^2} \sum_\alpha \operatorname{tr} \left[T \left(U_{\vec{x}_\perp,\alpha} i\overleftrightarrow{\partial}_- U^\dagger_{\vec{x}_\perp,\alpha} + U^\dagger_{\vec{x}_\perp - \hat{\alpha},\alpha} i\partial_- U_{\vec{x}_\perp - \hat{\alpha},\alpha} \right) \right] . \tag{2.10}$$

The Euler-Lagrange equations for $\vec{A}_+$ are

$$\partial_-^2 \vec{A}_+ = \frac{g}{a^2} \vec{J}_- \tag{2.11}$$

which contain no "time" derivatives $\partial_+ = \partial/\partial\tau$ and so can be solved for $\vec{A}_+$ in terms of $\vec{J}_-$ without upsetting our subsequent quantization. Eliminating $\vec{A}_+$ by (2.11) gives an effective action which only depends on the U's.

$$A = \int dx^+ dx^- \sum_{\vec{x}_\perp} \Big[\frac{1}{g^2} \sum_\alpha \operatorname{tr} \left(\partial_\mu U_{\vec{x}_\perp,\alpha} \partial^\mu U^\dagger_{\vec{x}_\perp,\alpha} \right) + \frac{1}{g^2 a^2} \sum_{\alpha\beta} \operatorname{tr} \left(U_{\vec{x}_\perp,\alpha} U_{\vec{x}_\perp + \hat{\alpha},\beta} U^\dagger_{\vec{x}_\perp + \hat{\beta},\alpha} U^\dagger_{\vec{x}_\perp,\beta} \right) + \int dx^{-\prime} \frac{g^2}{4a^2} |x^- - x^{-\prime}| \, \vec{J}_-(x^-) \cdot \vec{J}_-(x^{-\prime}) \Big] . \tag{2.12}$$

In order for this action to give the usual QCD action in the continuum it was necessary to assume that the matrices U have the form

$$U = e^{iag\vec{T}\cdot\vec{A}} \tag{2.13}$$

so that over small regions of space U can be expanded in a Taylor series. (This is the same as the "spin wave" expansion used at low temperature in statistical mechanics. In I it was incorrectly stated that in order for the expansion to be valid it was necessary for U to develop a vacuum expectation value. It is actually only necessary that over any small region of transverse space that the differences between U's be small.) Thus in particular we are assuming a functional measure for U which restricts it to the space of unitary matrices. There are several reasons why this is not the correct choice if we wish to describe physics correctly. The first and most mundane is that there is no compelling reason for finite values of the lattice spacing a to use this measure. Any measure which preserves the remaining gauge invariance under gauge transformations which are global in the longitudinal coordinates τ and z (i.e. a global SU(n) associated with each vertex of the transverse lattice) is a priori a suitable candidate. The correct measure can only be determined by a real space renormalization group analysis but might take the form

$$dU^\dagger dU e^{\int dx^+ dx^- V(a,U,U^\dagger)} \underset{a\to 0}{\longrightarrow} dU^\dagger dU \delta(U^\dagger U - 1)\,\delta(\det(U) - 1) \tag{2.14}$$

where U is allowed to be a complex general matrix, and dU is an unrestricted measure. This potential, V, may depend on the lattice spacing, a, and the invariants, $U^\dagger U$ and det(U). Put simply the variables of a lattice theory can never describe the underlying continuum theory exactly but are supposed to represent, in an aggregate way, the nearby degrees of freedom of the continuum theory. If the microscopic variables satisfy a constraint the new aggregate may not, and we may wish to choose a linear variable to describe the average behavior of a collection of nonlinear (i.e. unitary) variables. A more rigorous argument for the above statement comes from the, now exactly known,[13] behavior of the nonlinear (O(n)) σ-model in two dimensions. The local degrees of freedom are nonlinear, i.e. satisfy a constraint, and in greater than two dimensions the field would develop a vacuum expectation value and the spectrum would consist of (n-1) massless Goldstone bosons. In two dimensions the particle structure of the theory contains (n) massive scalars transforming under a linear realization of the symmetry group. Particle structure is inherently a large distance property and thus depends on the correct aggregate variables and not directly on the microscopic variables. Any physical description of the theory in terms of the underlying nonlinear degrees of freedom must be highly nonperturbative, but it is possible to achieve a much simpler description of the physics with an effective action in terms of explicitly linear variables and their interactions. For large n the form of the effective action for the O(n) nonlinear σ-model can be constructed explicitly.[14] In the action (2.12) the kinetic energy terms for U are identical to the action for an SU(n) $\otimes$ SU(n) nonlinear σ-model defined at each link of the transverse lattice. If the other two

terms of the lattice action which couple these σ-model systems together do not modify the above conclusions then the physically correct variables would be some linear variables to replace the U's as in (2.14). If we knew the exact behavior of the SU(n) × SU(n) σ-model we could rewrite the action (2.12) in terms of linear variables without approximation. As we don't, we adopt the ansatz (2.14) with a minimal set of operators included in V. The coefficients will have to remain as free parameters to be fit to the desired spectrum and be eventually determined through the continuum light analysis. It is useful to introduce the scalar field

$$M_{\vec{x}_\perp,\alpha} = \frac{1}{g} U_{\vec{x}_\perp,\alpha} \quad . \tag{2.15}$$

The effective potential takes the form

$$V = \mu_0^2 \mathrm{tr}(M^\dagger M) + \lambda_1 \mathrm{tr}((M^\dagger M)^2) + \lambda_2 \mathrm{tr}(M^\dagger M)^2 + \lambda_3 [\det(M) + \mathrm{h.c.}] + \ldots \quad . \tag{2.16}$$

Since in later chapters we will make use of the large N expansion only the first two terms in (2.16) will play any role in our quantitative analysis. Thus we can write the action in terms of M as

$$\begin{aligned} A = \int dx^+ dx^- \sum_{\vec{x}_\perp} \Bigg[& \sum_{\alpha\beta} \mathrm{tr}\left(\partial_\mu M_{\vec{x}_\perp,\alpha} \partial^\mu M_{\vec{x}_\perp,\alpha}\right) \\ & + \mu^2 \sum_\alpha \mathrm{tr}\left(M^\dagger_{\vec{x}_\perp,\alpha} M_{\vec{x}_\perp,\alpha}\right) + \lambda_2 \mathrm{tr}\left(\left(M_{\vec{x}_\perp,\alpha} M_{\vec{x}_\perp,\alpha}\right)^2\right) \\ & + \frac{g^2}{a^2} \sum_{\alpha\beta} \mathrm{tr}\left(M_{\vec{x}_\perp,\alpha} M_{\vec{x}_\perp+\hat{\alpha},\beta} M^\dagger_{\vec{x}_\perp+\hat{\beta},\alpha} M^\dagger_{\vec{x}_\perp,\beta}\right) \\ & + \int dx^{-\prime} \frac{g^2}{4a^2} |x^- - x^{-\prime}| \vec{J}_-(x^-)\cdot\vec{J}_-(x^{-\prime}) \Bigg] \quad . \end{aligned} \tag{2.17}$$

In this version of the action M is a full n x n complex matrix of fields.

We have discussed in some length the reasons why we have chosen the form (2.17) for the action of the theory. In return for a more physical starting point for the description of the physics, we have given up exact knowledge of some of the parameters which must be fit to the desired spectrum. We feel that there should exist an exact transformation to an effective linear theory derivable by renormalization group analysis but have not carried this out. The remaining task of this section is to quantize the action (2.17) in the light-cone. This is standard[15] and presents no new difficulties. The Hamiltonian corresponding to (2.17) is given by

$$P^{+} = H = \int dx^{-} \sum_{\vec{x}_\perp} \Bigg[\mu^2 \sum_{\alpha} \mathrm{tr}\left(M_{\vec{x}_\perp,\alpha} M^{\dagger}_{\vec{x}_\perp,\alpha} \right) -$$

$$- \frac{g^2}{a^2} \sum_{\alpha\beta} \mathrm{tr}\left(M_{x_\perp,\alpha} M_{x_\perp+\hat{\alpha},\beta} M^{\dagger}_{x_\perp+\beta,\alpha} M^{\dagger}_{x_\perp,\beta} \right) - \lambda \sum_{\alpha} \mathrm{tr}\left(\left(M_{\vec{x}_\perp,\alpha} M_{x_\perp,\alpha} \right)^2 \right)$$

$$- \int dx^{-\prime} \frac{g}{4a^2} \left| x^- - x^{-\prime} \right| \vec{J}_-(x^-) \cdot \vec{J}_-(x^{-\prime}) \Bigg] \quad . \tag{2.18}$$

The link fields may be decomposed into creation and annihilation operators in momentum space as

$$M_{\vec{x}_\perp,\alpha} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{2k} \left[A_{x_\perp,\alpha,k}\, e^{-ikx^-} + B^{\dagger}_{\vec{x}_\perp,\alpha,k}\, e^{+ikx^-} \right] \tag{2.19}$$

where the A's and B's obey the equal lightfront commutation relations

$$[A_k, A^{\dagger}_{k'}] = [B_k, B^{\dagger}_{k'}] = 2k\,\delta(k - k') \quad . \tag{2.20}$$

In the subsequent sections of this paper we will study the spectrum of the mass operator $\mu^2 = 2P^+P^-$ by diagonalizing it in the space generated by the decomposition (2.19) when limited to states containing few particles.

III. LONGITUDINAL DYNAMICS

In this paper we study the pure Yang-Mills sector of QCD based on the effective action of the previous section. Since the full theory cannot be solved we seek a practical and physically motivated approximation scheme. The principal ingredients of the scheme we propose are 1) a perturbative expansion in the terms in H responsible for transverse motion on the lattice, 2) a perturbative expansion in the nonlinearities introduced by the effective potential for the link mesons, 3) a topological expansion in the non-planarities of link meson interactions. The longitudinal free link meson dynamics and the longitudinal coulomb potential will be treated exactly. Although we performed a nonperturbative step by changing from the original gluonic degrees of freedom to the link mesons we have not yet constructed hadrons. In this section we shall study the transformation from the link meson gluons to intermediate hadronic degrees of freedom, the bare hadrons. This transformation is characterized by a single parameter and may have accordingly weak, strong or intermediate coupling features.

We start by showing how confinement arises in this approximation as a consequence of both symmetry and energetics: Using the transverse lattice and the light-cone gauge, the gluon dynamics have been separated into link mesons and coulomb potentials. The link mesons connect (x_+, x_-) sheets and interact in each sheet via coulomb potentials which are instantaneous and confined to the given sheet. All states of finite energy are singlets with respect to color rotations at each transverse site. It is this feature of local color confinement which leads to binding of the link mesons. The bare Hamiltonian which governs the longitudinal dynamics of the link mesons may be written as

$$H_0 = \sum_{\underset{\sim}{n},\alpha} \int dx\, \mu_0^2 \mathrm{tr}(M_{\underset{\sim}{n}\alpha} M^\dagger_{\underset{\sim}{n}\alpha})$$

$$- \frac{g^2}{4a^2} \sum_{n} \int dx dy\, |x-y| (\vec{J}_n(x)\cdot\vec{J}_n(y))^{\mathrm{Coul}} \qquad (3.1)$$

where

$$\vec{J}_{\underset{\sim}{n}} = \sum_\alpha \left(\mathrm{tr}\ \vec{\lambda}\, M_{\underset{\sim}{n}\alpha}\, i\overleftrightarrow{\partial}_-\, M^\dagger_{\underset{\sim}{n}\alpha} + \vec{\lambda}\, M^\dagger_{\underset{\sim}{n}-\hat{\alpha},\alpha}\, i\overleftrightarrow{\partial}_-\, M_{\underset{\sim}{n}-\hat{\alpha},\alpha} \right) \quad . \qquad (3.2)$$

The notation Coul. signifies that we are to keep only the direct coulomb potential part of the interaction and not those parts which produce or annihilate pairs. Using the expansion of M into plane wave creation and annihilation operators H_0 may be rewritten

$$H_0 = \sum_{\underset{\sim}{n},\alpha} \int_0^\infty \frac{dk}{2k} \frac{\mu_0^2}{2k}\, \mathrm{tr}\left(A^\dagger_{k,\underset{\sim}{n},\alpha} A_{k,\underset{\sim}{n},\alpha} + B^\dagger_{k,\underset{\sim}{n},\alpha} B_{k,\underset{\sim}{n},\alpha} \right)$$

$$- \frac{g^2}{4\pi a^2} \sum_{\underset{\sim}{n}} \int_{-\infty}^{\infty} \frac{dq}{q^2}\, \vec{J}_{\underset{\sim}{n}}(q)\cdot\vec{J}_{\underset{\sim}{n}}(-q) \qquad (3.3)$$

where

$$\vec{J}_{\underset{\sim}{n}}(q) = i \sum_\alpha \int_0^\infty \frac{dk}{4k(k+q)}\, \theta(k+q)(2k+q)\mathrm{tr}\Big(\vec{\lambda} : \Big((A_k A^\dagger_{k+q})_{n,\alpha}$$

$$- (B^\dagger_{k+q} B_k)_{\underset{\sim}{n},\alpha} - (A^\dagger_{k+q} A_k)_{\underset{\sim}{n}-\hat{\alpha},\alpha} + (B_k B^\dagger_{k+q})_{\underset{\sim}{n}-\hat{\alpha},\alpha} \Big) : \Big) \quad . \qquad (3.4)$$

In $J(q)$ pair operators have been dropped. The coulomb interaction represents a momentum transfer q from one link to another with a propagator $1/q^2$. The integral is infrared divergent when $q = 0$ and this divergence is regulated by a principal value prescription. It may be shown that the properties of color singlet states are independent of the way in which these infrared divergences are regulated.

By construction H_0 conserves the number of links or antilinks. The eigenstates of H_0 are thus all color singlet that can be initially classified according to the number of link mesons in each state. In this gauge the linked mesons may be ascribed the role of "valence gluons." The mass spectrum of each "number sector" is infinite and discrete due to the linear coulomb potential. H_0 also conserves the total longitudinal momentum and the group representation at each vertex, although we only consider states which are color singlets at each vertex. The simplest nontrivial sector of H_0 consists of one link and one antilink meson between the same pair of vertices. Such states are created by the action of $A^\dagger_{\underset{\sim}{n}\alpha} B^\dagger_{\underset{\sim}{n}\alpha}$ on the ground state. The only way to make a group singlet at both ends from this operator is to take the trace of the matrix product. Thus we consider the state

$$|P\rangle = \frac{1}{N} \int_0^1 dx\, \phi(x)(2x(1-x))^{-1/2} \mathrm{tr}\left(A^\dagger_{xP,\underset{\sim}{n},\alpha} B^\dagger_{(1-x)P,\underset{\sim}{n},\alpha} \right) |0\rangle \quad . \qquad (3.5)$$

The extra factors have been introduced so that the norm of this state is

$$< P | Q > \; = \; 2P\,\delta(P - Q) \tag{3.6}$$

if ϕ is normalized to

$$\int_0^1 dx\, |\phi(x)|^2 = 1 \quad . \tag{3.7}$$

Applying H_0 to this state gives

$$\begin{aligned} H_0\, |P> \; = \; & \frac{\mu_0^2}{2P} \frac{1}{N} \int_0^1 dx\, \phi(x)(2x(1-x))^{-1/2}(1/x + 1/(1-x))\,\mathrm{tr}\left(A^\dagger_{xP} B^\dagger_{(1-x)P} \right) |0> \\ & + \frac{1}{2P} \frac{2g^2 C_N}{r^2} \left(\fint_0^\infty \frac{dq(q+P)^2}{2q(q-P)^2} \right) \frac{1}{N} \int_0^1 dx\, \phi(x)(2x(1-x))^{-1/2}(1/x + 1/(1-x)) \\ & \mathrm{tr}\left(A^\dagger_{xP} B^\dagger_{(1-x)P} \right) |0> - \frac{1}{2P} \frac{2g^2 C_N}{\pi a^2} \int_0^1 \frac{dx}{x(1-x)} \frac{1}{N} \fint_0^1 dy\, \phi(y)(2y(1-y))^{-1/2} \\ & (x+y)(1-x) + (1-y)/(4\,|x-y|^2)\,\mathrm{tr}\left(A^\dagger_{xP} B^\dagger_{(1-x)P} \right) |0> \quad . \end{aligned} \tag{3.8}$$

To be an eigenstate $|P>$ must satisfy the equation

$$H_0\, |P> \; = \frac{M^2}{2P}\, | P > \quad . \tag{3.9}$$

Projecting out the momentum components gives the integral equation for ϕ

$$\begin{aligned} M^2 \phi(x) \; = \; & \left(\mu_0^2 + \frac{2g^2 C_N}{\pi a^2} \fint_0^\infty \frac{dq(q+P)^2}{q(q-P)^2} \right) \phi(x) \left(\frac{1}{x} + \frac{1}{1-x} \right) \\ & - \frac{2g^2 C_N}{\pi a^2} \fint_0^1 \frac{dx'}{|x - x'|^2}\, \phi(x')\, \frac{(x + x')((1-x) + (1-x'))}{4\sqrt{xx'(1-x)(1-x')}} \quad . \end{aligned} \tag{3.10}$$

In the first term on the right-hand side the quantity in brackets is the renormalized mass. This equation differs from the bound state equation obtained in the 't Hooft model[12] in two respects. First the current vertex of a boson introduces the additional spin factor $(x+x')((1-x)+(1-x'))/(4\sqrt{xx'(1-x)(1-x')})$ and second there is an overall factor of two since the links are bound by two coulomb potentials, one at each end.

We do not know an explicit form of the solution to this equation; nevertheless one can recognize some qualitative properties of the solutions. Since the equation is invariant under $x \leftrightarrow (1-x)$ with $\phi(x) \to \pm\phi(1-x)$, we may classify states as being even or odd with respect to this symmetry. $\phi(x)$ will have power behavior near $x \to 0$, or 1. If we suppose $\phi(x) \to x^\beta$ near $x \to 0$ we obtain the consistency condition

$$\mu^2 - \frac{g^2 C_N}{2\pi a^2} \fint_0^{1/x} dz\, \frac{z^{\beta - 1/2}(1+z)}{(1-z)^2} \; \xrightarrow[x \to 0]{} \; 0 \tag{3.11}$$

from the requirement that the leading singularity in x vanish. That the integral to converge at both ends requires $-\frac{1}{2} < \beta < \frac{1}{2}$. If $\beta > \frac{1}{2}$, the integral approaches a finite limit independent of x as $x \to 0$ so there is no solution. Performing the integral gives the condition on β

$$\mu^2 = \frac{2g^2 C_N}{\pi a^2}\, \pi\beta \tan \pi\beta \quad . \tag{3.12}$$

There are two solutions in the range $-\frac{1}{2} < \beta < \frac{1}{2}$ for any positive value of μ^2. There are no solutions for negative μ^2 which ultimately will imply that the lightest bare hadron is massive. For the limiting case $\mu^2 = \beta = 0$, $\phi(x) = 1$ fails to be a solution and the bound state mass remains finite. Thus at the bare hadron approximation, glueballs are inherently heavier than usual $\bar{q}q$ mesons. One could also note that the same feature holds in the strong coupling limit on the Wilson lattice and is a general property of the theory. An analysis of the self adjointness of H_0 rules out the negative solution for β. For highly excited states where the end point behavior is unimportant and this equation is effectively identical to the 't Hooft equation except for the previously mentioned factor of two, thus if one relates these bare hadrons to the spinless daughters of the Pomeron trajectory, the pomeron's trajectory would have half the slope of the "usual" particle trajectories. The parameter β determines the features of bare hadrons. For small β they are strongly bound states while for β near ½ they are weakly bound systems well described by two link mesons. The physical hadrons will eventually be labeled by a definite β.

In order to proceed with our program, we need to know the eigenvalues and eigenfunctions for the bound states in the two link mesons sector. A numerical procedure allows us to compute accurate values for the eigenvalues and eigenfunctions $\phi(x)$. In particular, we consider as a basis set of functions

$$\phi_n(x) = (x(1-x))^\beta P_n(x) \tag{3.13}$$

where $P_n(x)$ is an nth order polynomial in x which is either even or odd under $x \leftrightarrow (1-x)$. If the P_n are chosen as the appropriately normalized and scaled Jacobi polynomials the ϕ_n's provide an orthonormal basis. One may then diagonalize the finite dimensional matrix $H_{m,n} = \langle m | H_0 | n \rangle$. The eigenvalues provide estimates to the true eigenvalues which converge rapidly as the order of the matrix is increased. The eigenvectors which are the expansion coefficients of $\phi(x)$ in this

basis also provide converging estimates to the wave functions. The evaluation of the matrix elements can be performed analytically. First the polynomials are expanded as power series in $(x(1-x))^n$ or $(1-2x)(x(1-x))^n$ and then the integrals are performed using the identity

$$\int_0^1 dxdy \frac{x(1-x))^\alpha (y(1-y))^\beta}{|x-y|^2} = -\frac{2\pi\alpha\beta\Gamma(\alpha)\Gamma(\beta)}{2^{2\alpha+2\beta}(\alpha+\beta)\Gamma(\alpha+½)\Gamma(\beta+½)} \quad . \tag{3.14}$$

The convergence of $\phi(x)$ at the end points is only algebraic due to the presence of subdominant singularities. As these singularities are weak $(\sim x^{\beta+1}\ln(x))$, the convergence is nevertheless rapid. However, if we require higher derivatives of $\phi(x)$, the method would fail to converge. In all subsequent applications the matrix elements of ϕ which are needed are not highly sensitive to the end point behavior so this is not a restriction.

In Fig. 1 we plot the values of M^2 versus n the principal quantum number for $\beta = 0.1$. For fixed β, M^2 increases with n approaching linear dependence on n, for large n.

Returning to the whole lattice theory, the spectrum of H_0 in the two body sector consists of a set of coulomb bound states which may be labelled by a transverse coordinate $\vec{n}$, a polarization $\alpha = \hat{x}, \hat{y}$ indicating the orientation of the bound state with respect to $\vec{n}$, a quantum number n indicating the excitation of the state, and a total longitudinal momentum P. We may construct from this basis, states which are eigenstates of transverse momentum, but there would be no dependence of the energy on transverse momentum because of the local nature of H_0. This unphysical restriction we will approximately remove by the perturbative treatment of the remaining terms in H. To do so we must introduce the four body bare hadron states.

There are several distinct ways to place four links on the lattice so that group singlets may be formed at every vertex. These are indicated in Fig. 2. The fact that all the states should be color singlet at each transverse site leads to difficulties in defining the N_c gluon state at each transverse site leads to difficulties in defining the N_c gluon state in the limit $N_c \to \infty$; this is like the baryon problem for quarks and thus we do not discuss in this work three link meson states, etc. For the open boxes $B_{1,2}$ there is only one way of forming a group singlet which is to trace the group indices around the box. Proceeding as in the two body case we define a general state

$$|P\rangle = \frac{\sqrt{2}}{N^2} \int_0^1 dx_1 dx_2 dx_3 dx_4\, \delta(1 - x_1 - x_2 - x_3 - x_4)\phi(x_1, x_2, x_3, x_4)$$

$$(2x_1 2x_2 2x_3 2x_4)^{-1/2}\, \mathrm{tr}\left(A^\dagger_{x_1 P} A^\dagger_{x_2 P} B^\dagger_{x_3 P} B^\dagger_{x_4 P}\right)|0\rangle \qquad (3.15)$$

with the normalization condition

$$\int_0^1 dx_1 dx_2 dx_3 dx_4\, \delta(1 - x_1 - x_2 - x_3 - x_4)|\phi(x_1, x_2, x_3, x_4)|^2 = 1 \quad . \qquad (3.16)$$

Applying H_0 and projecting out momentum components leads to the bound state equation

$$M^2\phi = \mu^2\phi \sum_1^4 \frac{1}{x_i} + \left(N - \frac{1}{N}\right)(C_{12} + C_{23} + C_{34} + C_{41})\phi \qquad (3.17)$$

where

$$C_{ij}\phi = -\frac{g^2}{\pi a^2} \int dx_i' dx_j'\, \delta(x_i + x_j - x_i' - x_j') \frac{1}{|x_i - x_{i'}|^2}$$

$$\phi(\ldots, x_i', \ldots, x_j', \ldots)(x_i + x_i')/4\sqrt{x_i x_j x_i' x_j'} \quad . \qquad (3.18)$$

When four links impinge on a vertex there are two ways to make a group singlet. For example D_2 could be drawn in the two ways shown in Fig. 3. It turns out that for large N (the number of colors) these are the correct eigenstates, in group space, of H_0 with $|1\rangle$ a true four body bound state and $|2\rangle$ two independent two body bound states. For general N we must be more careful. The two orthogonal combinations are $|\pm\rangle = (|1\rangle \pm |2\rangle)/\sqrt{2N^3(N \pm 1)}$. In terms of these one obtains the bound state equation valid for the C and D configurations

$$M^2\phi^\pm = \mu^2\phi^\pm \Sigma \frac{1}{x_i} + (N - \tfrac{1}{N})(C_{14} + C_{23})\phi^\pm - \tfrac{1}{N}(C_{12} - C_{13} +$$

$$+ C_{14} + C_{23} - C_{24} + C_{34})\phi^\pm + \frac{N \pm 1}{2}(C_{12} + C_{14} + C_{23} + C_{34})\phi^\pm +$$

$$+ \frac{\sqrt{N^2 - 1}}{2}(C_{12} + C_{34} - C_{14} - C_{23})\phi^\mp \quad . \qquad (3.19)$$

In the limit $N \to \infty$ neglecting nonleading terms one obtains the two uncoupled equations

$$M^2\phi^1 = \mu^2\phi^1 \Sigma \frac{1}{x_i} + N(C_{12} + C_{23} + C_{34} + C_{41})\phi^1$$

$$M^2\phi^2 = \mu^2\phi^2 \Sigma \frac{1}{x_i} + 2N(C_{14} + C_{23})\phi^2 \quad . \qquad (3.20)$$

The equation for $|1\rangle$ is identical to the equation for the B configurations. The equation for $|2\rangle$ decouples the variables (x_1, x_4) from (x_2, x_3) and will reduce two independent two body states. For the E configurations, similar results are found.

If we adopt the approximation of keeping only the leading contribution in the 1/N expansion all of the possible configurations of four links on the lattice are described by the same bound state equation, or reduce to two independent two body bound states. As we shall see later in the same approximation only the four body bound states will contribute to the perturbative expansion of the transverse dynamics.

Many of the qualitative features of the two body equation apply to the four body equation as well. In particular the consistency condition that the most singular power cancel when each of the $x_i \to 0$ requires that $\phi(\ldots x_i \ldots) \to x_i^{\beta}$ as $x_i \to 0$, where β is the same as given in Eq. (3.12). The wave functions may also be classified according to their symmetry under exchanges of various indices since the bound state equation is invariant under cyclic permutations of the four variables or the reverse of their order. This is the symmetry of the dihedral group D_4 which has five irreducible representations four of which are one dimensional and one of which is two dimensional. Thus the spectrum will be characterized by five independent trajectories one of which will be populated by doublets. The WKB approximation to bound state equation is related to the spectrum of normal modes of a 3-simplex. Asymptotically the density of states only depends on the volume and dimensionality of the region and behaves as $n(M^2) \propto (M^2)^3$ as $M^2 \to \infty$, or M_n^2 behaves roughly as $n^{1/3}$.

The solutions of the bound state equation may be computed approximately just as in the two body case by constructing a variational basis

$$\phi_{\alpha,n}(x_1, x_2, x_3, x_4) = (x_1 x_2 x_3 x_4)^{\beta} P_n^{\alpha}(x_1, x_2, x_3, x_4) \tag{3.21}$$

where $x_4 = 1 - x_1 - x_2 - x_3$, and the P_n^{α} are chosen as linearly independent polynomials in the four variables which transform irreducibly with representation α under the group D_4. They are orthogonal with respect to α, but it is not particularly useful to require them to be orthogonal with respect to n since this is easily dealt with in the final numerical diagonalization. The eigenvalue equation is then cast into the form of a finite dimensional matrix eigenvalue problem in each sector separately by working out the necessary integrals analytically. This part of the calculation was greatly facilitated by the algebraic manipulation program MACSYMA. The resulting solutions to the matrix problems provide estimates to the masses and wave functions as before. The details of this calculation have been relegated to Appendix A.

Patterns of the radial excitations are now presented for our numerical calculations. The ground state energies of the various sectors (including the even and odd two-body sector) are plotted as a function of β in Fig. 4. Note that all ground state energies tend to some finite fixed value as $\beta \to 0$. In particular the smallest two-body eigenvalue approaches $\approx \pi^2/2$; the symmetric 4-link meson A1 seems to have a lower limiting value than the odd 2-link meson state. In Fig. 5 another look at the spectroscopy of bare hadrons is given. All the results can be generalized to include hadrons with a higher link meson number; however in the spirit of the approximation scheme one wants to explore the consequences of truncating the link mesons sector at some small number and at a latter stage check that indeed such an expansion is convergent. By now enough information about the bare hadrons has been accumulated and one may proceed to evaluate the full Hamiltonian in the Hilbert state of bare hadrons.

IV. TRANSVERSE DYNAMICS

The bare hadrons constructed in Sec. III from link mesons with coulomb binding are static on the transverse lattice. In the full Hamiltonian there are additional terms which generate nearest neighbor residual couplings enabling the various bare hadron states to move along the transverse lattice. They also break the conservation of link meson number mixing various n-link meson states. In particular, our calculation involves the mixing of two and four body bare hadron states. This effective field theory of bare hadrons is treated as a strong coupling perturbation expansion.

The various residual couplings can be identified by examining the full Hamiltonian governing the motion of the system. These terms are generated by the coulomb interaction, the magnetic interaction, and local potential interactions.

The coulomb terms are identified by expanding the current-current interaction, V^{CC}, in terms of creation and annihilation operators. We obtain

$$V^{CC} = \sum_n \frac{g^2}{4\pi a^2} \int \frac{dk}{k^2} \vec{J}_n(k) \cdot \vec{J}_n(k) \tag{4.1}$$

where the Fourier transform of the current is given by:

$$\begin{aligned} \vec{J}_n = \int_0^\infty \frac{dq}{\sqrt{2q}} \frac{dq'}{\sqrt{2q'}} \Big\{ & \delta(k-q-q')(q'-q) A^T_{n+\alpha,n}(q) B^T_{n+\alpha,n}(q') \\ & + \delta(k-q-q')(-q'-q) A^T_{n+\alpha,n}(q) B^\dagger_{n+\alpha,n}(q') \\ & + \delta(k+q-q')(q+q') A^\dagger_{n,n+\alpha}(q) B^T_{n,n+\alpha}(q') \\ & + \delta(k+q+q')(q-q') A^\dagger_{n,n+\alpha}(q) B^\dagger_{n+\alpha,n}(q') \Big\} \quad . \end{aligned} \tag{4.2}$$

In Fig. 6 we show the various vertices generated by this interaction, Figs. 6(a) and (b) show the coulomb scattering vertices, already used in the longitudinal dynamics calculations, which correspond to the terms

$$\frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k-q+q')(q+q') A^\dagger_{q'} A_q$$

$$\frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k-q+q')(-q-q') B^\dagger_{q'} B_q \quad .$$

The new vertices, Figs. 6(c) and (d), describe coulomb production and annihilation and are given by

$$\frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k+q+q')(q-q') A^\dagger_q B^\dagger_{q'}$$

$$\frac{1}{\sqrt{2q}} \frac{1}{\sqrt{2q'}} \delta(k-q-q')(q-q') A_q B_{q'} \quad .$$

The transverse magnetic term, which we call the non-local box, also gives rise to scattering and production terms as shown in Figs. 7(a) and (b). The scattering term (Fig. 7(a)) is

$$-\frac{g_{NLB}}{\pi a^2} \delta(u_q' + u_2' - u_1 - u_2) \frac{1}{\sqrt{2u_1}\sqrt{2u_2}\sqrt{2u_1'}\sqrt{2u_2'}}$$

and the production term (Fig. 7(b)) is

$$-\frac{g_{NLB}}{\pi a^2} \delta(u_1' - u_1 - u_2 - u_3) \frac{1}{\sqrt{2u_1'}\sqrt{2u_1}\sqrt{2u_2}\sqrt{2u_3}} \quad .$$

As noted in Section II, the linearization of the link meson degrees of freedom is only consistent if we add local potential terms to the effective Hamiltonian. One necessary term is a local four link meson interaction whose matrix elements are identical to those of the non-local box (replace g_{NLB} by g_{LB}). All the vertices are written with a normalization appropriate for direct matrix elements between the normalized wave functions $\phi(x_1,\ldots,x_n)$.

Our purpose is eventually to diagonalize the full Hamiltonian in a finite basis of bare hadrons. To this end we must calculate all the matrix elements connecting the various bare mesons which are given in terms of wave functions of the form

$$\phi(x_1, \ldots, x_N) = (x_1, \ldots, x_N)^{\beta} x_1^{n_1} \ldots x_N^{n_N} \text{ where } \sum_{i=1}^{N} x_i = 1 \quad .$$

Thus we wish to calculate analytically the various matrix elements of operators appearing in the Hamiltonian in this power basis.

The coulomb scattering matrix elements have already been calculated in Eq. (3.18) in addition to the kinetic energy. These interactions do not change the link meson number. Both local and nonlocal box scattering will have the same structure. Let us discuss in detail the nonlocal box case, which involves link mesons on four different sites. This interaction mixes the link meson number two and four sectors enabling the bare hadrons to move on the lattice. We note that a bare hadron containing two link mesons can move only via a second order transition through a four link meson state. The second effect of the coupling is to couple four link meson states with different "polarization" configurations. Thus four link meson states can move on the transverse lattice on their own without having to couple to the two or six link meson sector. An example of a process induced by the magnetic term is shown in Fig. 8, the process is denoted by $B(1, 1'2'3')\delta_{24}$, (for generality the number of spectators has been increased to N - 1) which is given by

$$B(123, 1')\delta_{42} = -g_{NLB}^2 \int du_1, \ldots, du_N \delta\left(1 - \sum_{i=1}^{N} u_i\right)$$

$$\int du_1' dy_1' dy_2' \delta(u_1' + y_1' + y_2' - u_1) \frac{1}{(2u_1' 2u_1 2y_1' 2y_2')^{1/2}}$$

$$\phi_{N+2}(u_1', y_1', y_2', u_2, \ldots, u_N) \phi_N(u_1, u_2, \ldots, u_N)$$

where

$$\phi_N(u_1, \ldots, u_N) = u_1^{\beta_1} u_2^{\gamma_2}, \ldots, u_N^{\gamma_N}$$

$$\phi_{N+2}(u_1', y_1', y_2', u_2, \ldots, u_N) = (u_1')^{\beta_1'} (y_1')^{\beta_2'} (y_2')^{\beta_3'} (u_2)^{\gamma_2'}, \ldots, (u_N)^{\gamma_N'} \quad ,$$

$$B(123, 1')\delta_{42}, \delta \ldots = \frac{1}{4} \frac{\Gamma(\beta_1' + 1/2)\Gamma(\beta_2' + 1/2)\Gamma(\beta_3' + 1)}{\Gamma(\beta_1' + \beta_2' + \beta_3' + \frac{3}{2})} g_{NLB}^2 \times$$

$$\frac{\Gamma(\beta_1 + \beta_1' + \beta_2' + \beta_3' + 1)\Gamma(\gamma_2 + \gamma_2' + 1) \ldots \Gamma(\gamma_N + \gamma_N' + 1)}{\Gamma(\beta_1 + \gamma_2 + \ldots \gamma_N + \beta_1' + \beta_2' + \beta_3' + \gamma_2' + \ldots \gamma_N' + N)} \quad . \tag{4.6}$$

All other matrix elements are calculated in an analogous manner. The (non)local box scattering relates two link meson states in the four link mesons number four sector. It is represented by Fig. 9 and its value is

$$\frac{(\beta_1'+\beta_2'+\beta_1+\beta_2+1)\Gamma(\beta_3'+\beta_3+1)\ldots\Gamma(\beta_N'+\beta_N+1)}{\Gamma(\beta_1'+\beta_2'+\ldots+\beta_N'+\beta_1+\ldots\beta_N+N-1)} \times$$

$$\frac{1}{4}\frac{\Gamma(\beta_1'+\tfrac{1}{2})\Gamma(\beta_2'+\tfrac{1}{2})\Gamma(\beta_1+\tfrac{1}{2})\Gamma(\beta_2+\tfrac{1}{2})}{\Gamma(\beta_1'+\beta_2'+1)\Gamma(\beta_1+\beta_2+1)} \quad . \tag{4.7}$$

Two other terms are induced by the current-current interaction. They are, respectively, the Coulomb annihilation term which is shown in Fig. 10. It relates states in the link meson four sector. The reason it does not appear in the link meson two sector is that it contains a colored gluon as the intermediate state in that case. It is given by

$$\frac{1}{4}\frac{\Gamma(\beta_1'+\beta_2'+\beta_1+\beta_2+1)\Gamma(\beta_3'+\beta_3+1)\ldots\Gamma(\beta_N'+\beta_N+1)}{\Gamma(\beta_1'+\beta_1+\ldots\beta_N'+\beta_N+N-1)} \times$$

$$\frac{\Gamma(\beta_1+\tfrac{1}{2})\Gamma(\beta_2+\tfrac{1}{2})\Gamma(\beta_1'+\tfrac{1}{2})\Gamma(\beta_2'+\tfrac{1}{2})}{\Gamma(\beta_1+\beta_2+2)\Gamma(\beta_1'+\beta_2'+2)} \times (\beta_1'-\beta_2')(\beta_1-\beta_2) \quad . \tag{4.8}$$

Note that this term is nonzero only between states that have the same parity under an exchange $u_1 \leftrightarrow u_2$ $(u_1' \leftrightarrow u_2')$. The last matrix element needed is the coulomb production matrix element. It relates the two and four link meson sectors, as shown in Fig. 11. It is given by

$$\frac{1}{4}\frac{\Gamma(\beta_1'+\beta_1+1)\ldots\Gamma(\beta_N'+\beta_N+1)}{\Gamma(\beta_1+\ldots\beta_N+\beta_1'+\gamma_1+\gamma_2+\beta_2'+\ldots\beta_N'+N)}\;\frac{(\gamma_1-\gamma_2)\Gamma(\gamma_1+\tfrac{1}{2})\Gamma(\gamma_2+\tfrac{1}{2})}{(\gamma_1+\gamma_2)(\gamma_1+\gamma_2+1)}$$

$$\left\{-\frac{\Gamma(\beta_1'+\tfrac{1}{2})\Gamma(\gamma_1+\gamma_2+\beta_1'+\beta_1+1)(\gamma_1+\gamma_2+2\beta_1'+1)}{\Gamma(\beta_1+\beta_1'+1)\Gamma(\beta_1'+\gamma_1+\gamma_2+\tfrac{3}{2})}\right. +$$

$$\left.\frac{\Gamma(\beta_1'+\tfrac{1}{2})\Gamma(\gamma_1+\gamma_2+\beta_2+\beta_2'+1)(\gamma_1+\gamma_2+2\beta_2'+1)}{\Gamma(\beta_2'+\beta_2+1)\Gamma(\beta_2'+\gamma_1+\gamma_2+\tfrac{3}{2})}\right\} \quad . \tag{4.9}$$

All matrix elements induced by the transverse current-current interaction have not been multiplied by the coulomb coupling constant $(g^2C_N)/\pi a^2$ which is set to be one. This coupling actually sets the mass scale of the problem. We have accumulated by now all the necessary information to calculate the masses of the glueballs. Having allowed the bare hadrons to move it would thus be natural to form a basis of transverse momentum eigenfunctions:

$$|\psi\rangle_{p_T} = \sum e^{iP_T\cdot na}|\psi n\rangle \tag{4.10}$$

and calculate the Hamiltonian in this basis. We have constructed the matrix for a general p_T along the lines described in this section. The full matrix is given in Appendix B.

Before turning to the actual computation of the hadron masses, we note that although we have restricted our discussion to link meson number not larger than four it is obvious that we have by now all the machinery necessary to deal with any link meson number. All one needs is to solve the n-link meson bare hadron wave function equation by the same techniques used for the four body wave function and then set up the Hamiltonian as done above. The number of terms increases rapidly of course, but the calculations are straightforward. This scheme is based on the proposition that the perturbation in the link meson number is indeed reasonable and provides a convergent procedure; this will be tested by the calculations.

V. CALCULATIONS

The theory as formulated consists of three parameters (including an overall scale) and an unknown function (the potential in the linear representation). This is a consequence of the theory being treated in a non-covariant gauge and with a non-covariant cut-off. The relation between parameters fixed by Lorenz invariance can only be recovered in the continuum limit. The fact that we have modified the theory for a finite lattice spacing was discussed in detail in Section II. This left us with the unknown potential function.

What we propose to check is that we have chosen relevant degrees of freedom in terms of which a tractable scheme to calculating hadron masses can be described. At our present state of knowledge this would require a parameter fitting to masses of known particles. We thus limit ourselves to describing features of this scheme.

The parameters are:

(1) $(g^2C_N)/\pi a^2$. Chosen to be one during the calculations, it sets the scale of bare hadron masses.

(2) γ^{*2} (or β)--The mass of the link meson (the edge point behavior of the n-link meson wave function). In the non-linear σ-model with O(N) symmetry this mass was actually generated dynamically. One would expect that in a covariant formulation there should be only one hadronic mass scale. The value of β controls the nature of the longitudinal dynamics. Large (small) β corresponds to weakly (strongly) bound link mesons.

(3) g_{NLB}--The coupling of the nonlocal box, in the continuum theory, it is essentially the gauge field self-coupling, g^2. For a fixed transverse lattice separation, a, it should be related to g^2. However in this calculation we treat it as a free parameter. $g^2_{NLB}/(\Delta E(\beta))$, where $\Delta E(\beta)$ are bare hadron energy splitting (which are a function of β), is essentially the expansion parameter of the effective bare hadron Hamiltonian.

(4) There are many other parameters associated with the effective potential containing $H_{n\alpha}$, all of which are in principle determined dynamically and calculable from g^2. We however do not know, at the moment, how to perform such a calculation and we thus pick one term--the local box interaction and treat the coupling g_{LB} as another free parameter.

Fixing three parameters one must next choose the basis of states and truncate them. In the calculations reported here we have chosen 3 even and 3 odd states in the link meson number two sector to each such classification. There correspond two possible "polarization" states (Fig. 2). In the four link meson sector, one state was chosen from each one-dimensional representation. There are four such representations (A1, A2, B1, B2) and to each correspond 10 different "polarization" states (Fig. 2). Two excitations were picked from the two-dimensional representation E. To each there correspond 8 states (E does not contain e1, e2 states). All together the basis contained 12+40+32 = 84 bare hadron states. The states were chosen by their energy ordering in the mass spectrum of bare hadrons.

We first studied the hadronic mass spectrum by diagonalizing the total Hamiltonian of zero transverse momentum. In this case the same symmetries that served to classify the n-link meson states, namely the bare hadrons, also categorize the eigenstates of the full Hamiltonian. For an illustrative example let us turn to the link meson number two sector. In that case the states A_1 and A_2 can form only three of the five representations of D_4.

Taking the symmetric and antisymmetric combinations of even two body states one forms the one-dimensional representations of A1 and B2

$$|A1> = \frac{1}{\sqrt{2}}(|A_1>^{even} + |A_2>^{even})$$

$$|B2> = \frac{1}{\sqrt{2}}(|A_1>^{even} - |A_2>^{even}) \quad .$$

The odd two-body wave function can be placed in a doublet to form the two-dimensional E representation

$$|E> = \begin{pmatrix} |A_1>_{odd} \\ \pm |A_2>_{odd} \end{pmatrix} \quad .$$

Both the A2 and B1 representations are absent from the link meson number two sector. In order to appreciate this fact let us consider a spin ℓ particle on a transverse lattice. (These results are valid also for helicity states in the IMF.) The irreducible representations of dimension $2\ell + 1$ will be broken into one and two-dimensional representations on the transverse lattice. By applying standard methods of group theory one concludes that the spin zero state transforms like A1. The $\ell_z = 0$ component of the spin one state is in A1 while the $\ell_z = \pm 1$ components form two-dimensional E representations. In the spin two case the $\ell_z = 0$ component transforms like A1. The $\ell_z = \pm 1$ are in an E representation and the $\ell_z = \pm 2$ form symmetric and antisymmetric combinations which transform according to B1 and B2 respectively. The general rule is:

$$\left.\begin{matrix} A1 \\ A2 \end{matrix}\right\} \ell_z = \pm 4, \pm 8, \pm 12 \quad \begin{matrix} m = 0 \text{ symmetric} \\ \text{antisymmetric} \end{matrix}$$

$$\left.\begin{matrix} B1 \\ B2 \end{matrix}\right\} \ell_z = \pm 2, \pm 6, \pm 10 \quad \begin{matrix} \text{antisymmetric} \\ \text{symmetric} \end{matrix}$$

$$E \} \quad \ell_z = \pm 1, \pm 3, \pm 5 \quad .$$

In the continuum limit one should recover the Lorenz degeneracy. Thus the absence of the B1 and A2 representations in the two link meson sector means that these would have to come from the higher link meson number sector. This implies that the two and four (etc.) sectors have to mix in the continuum limit.

Note that one knows already from the study of the bare hadrons in the link meson number two sector that the lowest even eigenstate is lighter, for all β, than the lowest odd eigenstate. This leads to degenerate A1, B2 states split away from an E state, identifying the A1 state with a spin zero hadron, B2 with an $m = \pm 2$ symmetric component of a spin two component and E with $m = \pm 1$. We get an embryonic degeneracy between a scalar and a tensor piece of a glueball. If E will also be related to the tensor then the vector glueball lies way above the scalar and some components of the tensor. Even in the case that the E will be related to a vector particle a hierarchy scalar-tensor and a heavier vector has formed.

A similar classification can be obtained for the link meson number four sector at $k = 0$. These symmetry considerations serve also as a check of our computer program. Most of the eigenfunctions have been checked to show that indeed all eigenfunctions of the total Hamiltonian can be classified according to D4 and that they contain only the allowed combinations in the two and four link meson number scalar. The classification of all four body states is shown in Appendix C.

In Fig. 12 we show the mass spectrum for two sets of parameters:

a. $\beta = 0.2$, $g^2_{NLB} = -3.65$, $g^2_{LB} = 0$

b. $\beta = 0.45$, $g^2_{NLB} = -46$, $g^2_{LB} = 0$.

These lower lying states are classified according to the D_4 symmetry.

The next question to be addressed is the consistency of the link number expansion. It is checked by studying the four body content of the low-lying excitations. These results are summarized in Fig. 12. It turns out that the expansion is rather satisfactory. The four body mixing in all possible cases (A1, B2 and E) is not too large as to upset the expansion and not too small, thus allowing the two-body states to move on the transverse lattice. (For the set of values (a) perturbation theory would give a 50% error while for the set of values (b) although g^2_{NLB} is very large, perturbation theory gives a correct result within 10% because $g^2_{NLB}/\Delta E(\beta)$, the expansion parameter, is small.)

In the absence of a physical glueball spectrum qualitative considerations are used to zone the relevant range of parameters. For a general set of parameters the Hamiltonian is not positively definite. The requirement that all bound states have positive energies constitutes a non-trivial constraint. In particular, for $g^2_{LB} = 0$, a curve forms in the β, g^2_{NLB} plane. One defines g^2_{max} to be the largest absolute value of g^2_{NLB} allowed for a fixed value of β. g^2_{max} is an increasing function of β. Its behavior is shown in Fig. 13. In the allowed region of the plane one looks for those parameters which simulate on a rather large lattice the continuum limit.

The A1-B2 degeneracy is broken once the full Hamiltonian is diagonalized leading to a scalar-tensor-vector like ordering. This result is stable under a large variation of all the parameters of the theory and is one of the qualitative results of our analysis. One should point out that in the calculation of Kogut-Susskind a scalar-tensor degeneracy is obtained (or more precisely: in a cubic lattice a spin two breaks into one three and one two dimensional representations the statement is that the scalar is degenerate with the three dimensional piece of the tensor) and they are both lighter than the vector (all of whose components fall in a single three

dimensional representation on the cubic lattice and thus remain degenerate). As a matter of fact two interacting spin 1 particles with a conventional potential would be in an s-wave and thus if the angular momentum interactions are not too large, one indeed should have the vector lie above the scalar and tensor.

A naive guess could be that having a light link meson mass μ^2--which is tantamount to β near zero (strong longitudinal coupling)--would pull together the bare hadron masses and help them mix. However as was shown in Sec. III by reducing β to zero one cannot generate even one zero mass hadron (unlike the situation in the bare quark-antiquark meson sector). The ground states of the various representations become increasingly heavier because of the "welding energy" at each transverse site. This a priori spacing of the low lying bare particles means that a non-negligible coupling should exist between two- and four-body states, in particular g_{NLB}, would have to be large. However for small β the lowest lying even two-body state has a small mass relative to its separation from the four body states. Thus a large g_{NLB} falls outside the allowed region. We are thus pushed to values of β around 0.2 and $g_{NLB} \simeq -3.5$ (the values used in Fig. 12) before a reasonable four-body mixing occurs. Even in this range some improvement may be desired. The energy difference between the bare two-body even state and the first four-body B2 state is rather large and it mixes much more into the second excited two-body even state only for $\beta \simeq 0.45$. g_{NLB} can be made large enough to overcome the energy difference and reach a 20% mixture. This reflects itself in the dispersion relations for non-zero transverse momentum.

Before studying the $k_T \neq 0$ case in detail we note that the local box term has a negative eigenvalue and thus cannot serve to increase significantly the allowed region. By studying the structure of theory at $k_T \neq 0$ one gets additional information on the "continuum like" behavior of the excitations. The Hamiltonian described in the former section is diagonalized for nonzero transverse momentum. In Fig. 14 we plot the dispersion relation: the energy of the hadrons vs. their transverse momentum. The x and y transverse momentum components are equal.

We first note that each of the lowest lying states involving two and four link meson mixing satisfies a physical acoustic dispersion. This allows an appropriate transverse motion for the hadrons. The next information available relates to the extent to which the hadrons in this approximation have continuum like properties. We can study several aspects of this question; in the infinite momentum frame the dispersion relation is

$$E = m + ck_T^2$$

where c stands for the velocity of light. In the continuum limit all states should have the same coefficient c and in addition the rotational invariance of the theory should be restored. In our case one compares c in the x or y direction to c in the 45° direction ($k_x = k_y$). In Table I we list c for the lowest lying A1, B2 and one of the E states (for $k_T \neq 0$ the E states are no longer exactly degenerate). The large β and g_{NLB} region is again more continuum like. For the 45° direction c is the same for all representations within 20%. For A1 and B2 states the 45° rotation changes c by less than 10%. The E states are not yet behaving in a satisfying manner. One state (shown in the table) has a very large value of c and the other (not shown) is essentially flat (they change roles when going from $(k_x, 0)$ to $(0, k_y)$).

For smaller values of β the situation is somewhat different for B2 states, which hardly mix with the high energy four body state. As a final observation we note that for large β both the A1 and B2 masses are much smaller than the value of

the energy for large k_T. This is in the right direction as the edge of the Brilouin zone moves to infinity for small values of the lattice spacing.

VI. DISCUSSION

The transverse lattice, infinite momentum frame version of QCD attempts to deal directly with the physical degrees of freedom of QCD, maintaining and imposing the full internal symmetry structure of the theory at the cost of the full Lorentz symmetry. The main thrust of this paper is in actually implementing this program. This was done by treating the longitudinal dynamics nonperturbatively and perturbing in the transverse motion dynamics.

The hadrons emerging from the analysis are composed of a superposition of bare hadrons. Each bare hadron is a weakly bound system of link mesons (β is rather large). Link meson number violation is large enough to allow a reasonable transverse motion but is small enough to validate the expansion in terms of link meson number. Strongly bound bare hadrons cannot be supported by our approximation to the effective potential. An improvement of our understanding of the linear version is needed before it can be established as a faithful effective theory at some hadronic distance scale.

We wish to conclude with some remarks on the general characteristics of the glueball spectrum. Glueballs have the rather unique property of being formed from gauge particles which are color non-singlets but have triality zero. One would like to know in what way will the glueball spectrum reflect these special facets of its constituents.

In our analysis the Lorentz pattern of the low lying excitations seems to be explained by a valence gluon picture. It is however not clear that this description is faithful. The first source of doubt is the gauge dependence of the separation between the dependent potential and the independent degrees of freedom. However if there exists any gauge in which a valence gluon picture serves to classify the gauge invariant physical states, it is a useful concept. The more serious problem is the strong coupling like bias inherent in our analysis. In both the straightforward strong coupling calculations in the $A^0 = 0$ gauge[7] and in our more complicated approach the valence structure is a strong coupling feature. The expansion in link meson number dictates (as was shown in Section V) the order of the Lorentz excitations.

To leading order of the $1/N_c$ expansion an infinite number of stable glueballs was obtained. This is consistent with expectations[10] from the continuum limit. In this sense there exists a limit in which gluons (link mesons) are confined and attract with constant forces. In our scheme there was no algebraic characterization of the states save that they are color singlets. One may wonder if these states form some degeneracy patterns associated with a surviving global symmetry. Such a symmetry should be explicit in a string theory of hadrons.

APPENDIX A. SOME DETAILS OF THE SOLUTION OF THE FOUR BODY BOUND STATE EQUATION

The four body bound state equation is simplified when decomposed in terms of states with a definite D_4 symmetry. D_4 has four different one dimensional representations A1, A2, B1, B2 and one two dimensional representation E. The character table of the group D_4 can be found in standard books on finite dimensional groups.[11] It is:

	identity	C_2	$2C_4$	$2C_2'$	$2C_2''$
A1	1	1	1	1	1
A2	1	1	1	-1	-1
B1	1	1	-1	1	-1
B2	1	1	-1	-1	1
E	2	-2	0	0	0

where the various group members of D_4 generate the following transformations on square whose sides are denoted (in order) by x, y, z, w.

C_2	$x \leftrightarrow z \quad y \leftrightarrow w$
$C_4(90^\circ)$	$xyzw \rightarrow yzwx$
$C_4(270^\circ)$	$xyzw \rightarrow wxyz$
$C_{2'I}$	$xyzw \rightarrow wzyx$
$C_{2'II}$	$xyzw \rightarrow yxwz$
$C_{2''I}$	$xyzw \rightarrow zyxw$
$C_{2''II}$	$xyzw \rightarrow xwzy$

In the two-dimensional E representation one has

$$C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_4^{90^\circ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad C_4^{270^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C_{2'I} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad C_{2'II} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C_{2''I} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C_{2''II} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The following trial functions were constructed as irreducible representations of the various representations (the constraint $x + y + z + w = 1$ was imposed). The matrix elements of the kinetic energy and the potential energy were calculated using Eq. (3.14) and the algebra was kept under control with the help of MACSYMA.

APPENDIX B. THE HAMILTONIAN FOR GENERAL TRANSVERSE MOMENTUM

The Hamiltonian is given in the basis of two and four body states as defined in Fig. 2. A dictionary for the symbols is given below and an example was done in Eq. (4.5).

$$\phi_{\alpha,n}(x_1, x_2, x_3, x_4) = (x_1 x_2 x_3 x_4)^\beta P_n^\alpha(x_1, x_2, x_3, x_4) \quad . \qquad (3.21)$$

For $\alpha = A1$

$$\begin{aligned} P_0 &= 1 \\ P_1 &= x^2 + y^2 + z^2 + w^2 + xy + yz + zw + wx \\ P_2 &= x^3 + y^3 + z^3 + w^3 \\ P_3 &= x^4 + y^4 + z^4 + w^4 + x^2z^2 + y^2w^2 + x^3z + zx^3 + y^3w + w^3y + xyzw \quad . \end{aligned} \qquad (A.1)$$

For $\alpha = A2$

$$P_0 = xy(y - x) + yz(z - y) + zw(w - z) + wx(x - w) \quad . \qquad (A.2)$$

For $\alpha = B1$

$$P_0 = (x - z)(y - w) \quad . \qquad (A.3)$$

For $\alpha = B2$

$$\begin{aligned} P_0 &= x + z - (y + w) \\ P_1 &= x^2 + z^2 - (y^2 + w^2) \\ P_2 &= x^3 + z^3 - (y^3 + w^3) \\ P_3 &= (x + z)yw - (y + w)xz \quad . \end{aligned} \qquad (A.4)$$

For $\alpha = E$

$$\begin{aligned} P_0 &= (x - z, y - w) \\ P_1 &= (x^2 - z^2, y^2 - w^2) \\ P_2 &= (x^3 - z^3, y^3 - w^3) \\ {}_1P_3 &= (yw(x - z), xz(y - w)) \\ {}_3P_2 &= (xz(x - z), yw(y - w)) \quad . \end{aligned} \qquad (A.5)$$

M = kinetic energy

C = direct coulomb

LB = local box

B = nonlocal box

CP = coulomb production

CA = coulomb annihilation

$e_x = e^{ik_x a}$

$e_y = e^{ik_y a}$

	A_1	A_2
A_1	$M_1 + M_2 + 2C_{12} + LB_{12}$	0
A_2	0	$M_1 + M_2 + 2C_{12} + LB_{12}$
B_1	$B(234, 2')\delta_{11'} + e_y B(412,1')\delta_{32'}$	$B(123,1')\delta_{42'} + e_x B(341,2')\delta_{21'}$
B_2	$B(123,1')\delta_{42'} + e_y B(341,2')\delta_{21'}$	$B(234,2')\delta_{11'} + e_x B(412,1')\delta_{32'}$
C_1	$CP(1234, 1'2')$	$e_y CP(3412, 2'1')$
C_2	$CP(3412, 2'1')$	$CP(1234, 2'1')$
C_3	$e_x CP(3412, 1'2')$	$e_y CP(1234, 1'2')$
C_4	$e_x CP(1234, 2'1')$	$CP(3412, 1'2')$
D_1	$CP(1234,1'2') + e_y CP(3412,2'1')$	0
D_2	0	$CP(1234,1'2') + e_x CP(3412,2'1')$
E_1	0	$\sqrt{2}CP(1234,1'2') + \sqrt{2}CP(4123,2'1') + \sqrt{2}LB(123,1')\delta_{42'} + \sqrt{2}LB(234,2')\delta_{11'}$
E_2	$\sqrt{2}CP(1234,1'2') + \sqrt{2}CP(4123,2'1') + \sqrt{2}LB(123,1')\delta_{42'} + \sqrt{2}LB(234,2')\delta_{11'}$	0

	B_1	B_2
A_1	$B(2,2'3'4')\delta_{11'} + e_x^* B(1,4'1'2')\delta_{23'}$	$B(1,1'2'3')\delta_{24'} + e_x^* B(2,3'4'1')\delta_{12'}$
A_2	$B(1,1'2'3')\delta_{24'} + e_y^* B(2,3'4'1')\delta_{12'}$	$B(2,2'3'4')\delta_{11'} + e_y^* B(1,4'1'2')\delta_{23'}$
B_1	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$	0
B_2	0	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$
C_1	$B(34,3'4')\delta_{11'}\delta_{22'}$	$B(12,1'2')\delta_{33'}\delta_{44'}$
C_2	$B(34,2'3')\delta_{21'}\delta_{14'}$	$B(12,2'3')\delta_{34'}\delta_{41'}$
C_3	$B(34,4'1')\delta_{12'}\delta_{23'}$	$B(12,4'1')\delta_{32'}\delta_{43'}$
C_4	$B(34,1'2')\delta_{24'}\delta_{13'}$	$B(12,3'4')\delta_{31'}\delta_{42'}$
D_1	0	0
D_2	0	0
E_1	0	0
E_2	0	0

	C_1	C_2
A_1	$CP(12,1'2'3'4')$	$CP(21,3'4'1'2')$
A_2	$e_y^*(21,3'4'1'2')$	$CP(21,1'2'3'4')$
B_1	$B(34,3'4')\,\delta_{11'},\delta_{22'}$	$B(23,\ 3'4')\,\delta_{12'}\,\delta_{41'}$
B_2	$B(12,1'2')\,\delta_{33'},\delta_{44'}$	$B(23,1'2')\,\delta_{14'},\delta_{43'}$
C_1	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$ $+LB_{23}+LB_{41}+CA_{23}+CA_{41}$	$e_y CA(41,2'3')\,\delta_{31'},\delta_{24'}$
C_2	$e_y^* CA(23,4'1')\,\delta_{13'},\delta_{42'}$	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$ $+LB_{23}+LB_{41}+CA_{23}+CA_{41}$
C_3	$e_x CA(41,2'3')\,\delta_{31'},\delta_{24'}$	0
C_4	0	$e_x CA(23,4'1')\,\delta_{13'}\delta_{42'}$
D_1	$CA(23,2'3')\,\delta_{11'},\delta_{44'}$	$e_y CA(41,4'1')\,\delta_{33'},\delta_{22'}$
D_2	$e_x e_y^* CA(41,4'1')\,\delta_{22'},\delta_{33'}$	$e_x CA(41,2'3')\,\delta_{24'},\delta_{31'}$
E_1	$e_y^* \sqrt{2} CA(41,4'1')\,\delta_{22'},\delta_{33'}$	$\sqrt{2} CA(23,2'3')\,\delta_{11'},\delta_{44'}$
E_2	$\sqrt{2} CA(23,2'3')\,\delta_{11'},\delta_{44'}$	$\sqrt{2} CA(34,4'1')\,\delta_{12'}\delta_{23'}$

	C_3	C_4
A_1	$e_x^* CP(12,3'4'1'2')$	$e_x^* CP(21,1'2'3'4')$
A_2	$e_y^* CP(12,1'2'3'4')$	$CP(12,3'4'1'2')$
B_1	$B(41,34)\,\delta_{21'},\delta_{32'}$	$B(12,3'4')\,\delta_{31'},\delta_{42'}$
B_2	$B(41,1'2')\,\delta_{23'},\delta_{34'}$	$B(34,1'2')\,\delta_{13'},\delta_{24'}$
C_1	$e_x^* CA(23,4'1')\,\delta_{13'},\delta_{42'}$	0
C_2	0	$e_x^* CA(41,2'3')\,\delta_{24'},\delta_{31'}$
C_3	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$ $+LB_{23}+LB_{41}+CA_{23}+CA_{41}$	$e_y CA(23,4'1')\,\delta_{13'},\delta_{42'}$
C_4	$e_y^* CA(41,2'3')\,\delta_{31'},\delta_{24'}$	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$ $+LB_{23}+LB_{41}+CA_{41}+CA_{23}$
D_1	$e_x^* CA(23,4'1')\,\delta_{13'},\delta_{42'}$	$e_x^* e_y CA(41,2'3')\,\delta_{31'},\delta_{24'}$
D_2	$e_y^* CA(23,2'3')\,\delta_{11'},\delta_{44'}$	$CA(23,4'1')\,\delta_{13'},\delta_{42'}$
E_1	$e_y^* \sqrt{2} CA(12,2'3')\,\delta_{34'},\delta_{41'}$	$\sqrt{2} CA(12,4'1')\,\delta_{32'},\delta_{43'}$
E_2	$e_x^* \sqrt{2} CA(41,4'1')\,\delta_{22'},\delta_{33'}$	$e_x^* \sqrt{2} CA(34,2'3')\,\delta_{14'},\delta_{21'}$

	D_1	D_2
A_1	$CP(12,1'2'3'4') + e_y^* CP(21,3'4'1'2')$	0
A_2	0	$CP(12,1'2'3'4') + e_x^* CP(21,3'4'1'2')$
B_1	0	0
B_2	0	0
C_1	$CA(23,2'3')\delta_{11'}\delta_{44'}$	$e_x^* e_y CA(41,4'1')\delta_{22'}\delta_{33'}$
C_2	$e_y^* CA(41,4'1')\delta_{33'}\delta_{22'}$	$e_x^* CA(23,4'1')\delta_{42'}\delta_{13'}$
C_3	$e_x CA(41,2'3')\delta_{31'}\delta_{24'}$	$e_y CA(23,2'3')\delta_{44'}\delta_{11'}$
C_4	$e_x e_y^* CA(23,4'1')\delta_{42'}\delta_{13'}$	$CA(41,2'3')\delta_{31'}\delta_{24'}$
D_1	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$ $+LB_{23}+LB_{41}+CA_{23}+CA_{41}$	0
D_2	0	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}+C_{41}$ $+LB_{23}+LB_{41}+CA_{23}+CA_{41}$
E_1	0	$\sqrt{2}CA(23,3'4')\delta_{41'}\delta_{12'}+e_x\sqrt{2}CA(41,2'3')\delta_{31'}\delta_{24'}$
E_2	$\sqrt{2}CA(23,2'3')\delta_{11'}\delta_{44'}$ $+e_y^*\sqrt{2}CA(34,4'1')\delta_{12'}\delta_{23'}$	0

	E_1	E_2
A_1	0	$\sqrt{2}CP(12,1'2'3'4')+\sqrt{2}CP(21,4'1'2'3')$ $+\sqrt{2}LB(1,1'2'3')\delta_{24'}+\sqrt{2}LB(2,2'3'4')\delta_{11'}$
A_2	$\sqrt{2}CP(12,2'3'4'1')+\sqrt{2}CP(21,1'2'3'4')$ $+\sqrt{2}LB(1,4'1'2')\delta_{23'}+\sqrt{2}LB(2,3'4'1')\delta_{12'}$	0
B_1	0	0
B_2	0	0
C_1	$e_y\sqrt{2}CA(41,4'1')\delta_{22'}\delta_{33'}$	$\sqrt{2}CA(23,2'3')\delta_{12'}\delta_{44'}$
C_2	$\sqrt{2}CA(23,2'3')\delta_{11'}\delta_{44'}$	$\sqrt{2}CA(41,3'4')\delta_{21'}\delta_{32'}$
C_3	$e_y\sqrt{2}CA(23,1'2')\delta_{43'}\delta_{14'}$	$e_x\sqrt{2}CA(41,4'1')\delta_{22'}\delta_{33'}$
C_4	$\sqrt{2}CA(41,1'2')\delta_{23'}\delta_{34'}$	$e_x\sqrt{2}CA(23,3'4')\delta_{41'}\delta_{12'}$
D_1	0	$\sqrt{2}CA(23,2'3')\delta_{11'}\delta_{44'}+e_y\sqrt{2}CA(41,3'4')\delta_{21'}\delta_{32'}$
D_2	$\sqrt{2}CA(23,3'4')\delta_{41'}\delta_{12'}$ $+e_x\sqrt{2}CA(41,2'3')\delta_{31'}\delta_{24'}$	0
E_1	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}$ $+C_{41}+LB_{12}+LB_{23}+LB_{34}+LB_{41}$ $+CA(12)+CA(23)+CA(34)+CA(41)$	0
E_2	2	$M_1+M_2+M_3+M_4+C_{12}+C_{23}+C_{34}$ $+C_{41}+LB_{12}+LB_{23}+LB_{34}+LB_{41}$ $+CA(12)+CA(23)+CA(34)+CA(41)$

APPENDIX C. DECOMPOSITION OF HADRONS, AT ZERO TRANSVERSE MOMENTUM, IN TERMS OF BARE HADRON SYMMETRIES

One can classify the hadrons with $k_T = 0$ according to the symmetry D_4. However only certain configurations of bare hadrons are allowed in a hadron of a given symmetry. The following is a list of such a decomposition. The letters a,b,c,d,e correspond to the states A_1, A_2, B_2, B_2, C_1, C_2, C_3, C_4, D_1, D_2, E_1, E_2 appearing in Fig. 2.

<u>A1</u>: Even $\{a(++)\}$, A1 $\{b(++), c(++++), d(++), e(++)\}$, A2 $\{b(++), c(++++), d(++), e(++)\}$, B1 $\{c(++++), d(++)\}$, B2 $\{c(++++), d(++)\}$

<u>A2</u>: A1 $\{b(+-)\}$, A2 $\{b(+-)\}$, E(1) $\{c(++++)\}$, E(2) $\{c(++++)\}$

<u>B1</u>: A1 $\{c(+--+)\}$, A2 $\{c(+--+)\}$, B1 $\{b(++), c(+--+)\}$, B2 $\{b(++), c(+--+)\}$

<u>B2</u>: Even $\{a(+-)\}$, A1 $\{d(+-), e(+-)\}$, A2 $\{d(+-), e(+-)\}$, B1 $\{b(+-), d(+-)\}$, B2 $\{b(+-), d(+-)\}$, E(1) $\{c(+--+)\}$, E(2) $\{c(+--+)\}$

<u>E</u>: $\text{Odd}\begin{Bmatrix} a1 \\ a2 \end{Bmatrix}$, $\text{A1}\left\{\begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix}\right\}$, $\text{A2}\left\{\begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix}\right\}$,

$$\text{B1}\left\{\begin{pmatrix} e_2 \\ -e_1 \end{pmatrix}, \begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix}\right\}, \text{B2}\left\{\begin{pmatrix} e_2 \\ -e_1 \end{pmatrix}, \begin{pmatrix} c(-+-+) \\ c(++--) \end{pmatrix}\right\},$$

$$\begin{pmatrix} E(1)b1+E(2)b2 \\ E(2)b1+E(1)b2 \end{pmatrix}, \begin{pmatrix} E(1)b2-E(2)b1 \\ E(1)b1-E(2)b2 \end{pmatrix},$$

$$\text{E1}\left\{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} c(++--) \\ c(+-+-) \end{pmatrix}\right\}, \text{E2}\left\{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} c(++--) \\ c(+-+-) \end{pmatrix}\right\}$$

This assumes the following sign conventions for the E states: E(1)(1234) = E(2)(2341) = -E(2)(4123) and E(2)(1234) = -E(1)(2341) = E(1)(4123).

Table I. The velocity of light of various hadrons. The units of C are determined by $(g^2 C_N)/(\pi a^2) = 1$.

$\beta = 0.2$; $g^2_{NLB} = -3.65$; $g_{LB} = 0$

	$C_{k_x = k_y}$	$C_{k_x, k_y = 0}$
A1	23	22
B2	5	6
E	19	36

$\beta = 0.45$; $g^2_{NLB} = -46$; $g_{LB} = 0$

	$C_{k_x = k_y}$	$C_{k_x, k_y = 0}$
A1	145	135
B2	167	177
E	135	270

FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

Fig. 1: The masses of two body even (2E) (marked (x)) and odd (2O) (marked (X)) bare hadrons as a function of the radial excitation index, n. Plots are for $\beta = 0.1$.

Fig. 2: Two and four body lattice configurations.

Fig. 3: Possible color structure of the D_2 four body state.

Fig. 4: Ground state energies of two and four body bare hadrons is plotted as a function of β.

Fig. 5: Bare hadron mass spectrum for $\beta = 0.1$.

Fig. 6: The vertices generated by the coulomb terms in the current-current interaction. Figs. (a) and (b) are the coulomb scattering terms. Figs. (c) and (d) represent coulomb production and annihilation respectively.

Fig. 7: Scattering (Fig. (a)) and production (Fig. (b)) terms resulting from the transverse magnetic interaction.

Fig. 8: The process denoted by $B(1,1'2'3')\,\delta_{24'}$.

Fig. 9: The nonlocal box scattering term.

Fig. 10: Coulomb annihilation terms.

Fig. 11: Coulomb production terms.

Fig. 12: The hadronic mass spectrum for (a) $\beta = 0.2$, $g^2_{NLB} = -3.65$, $g_{LB} = 0$; (b) $\beta = 0.45$, $g^2_{NLB} = -46$, $g_{LB} = 0$. The bare hadron masses are also shown. The arrowed lines indicate the main bare hadron wave function decomposition of the hadrons. The number is the mixing probability (in %). For $\beta = 0.45$ only the important two body states were marked.

Fig. 13: The maximum g^2_{NLB} for a given β which leads to positive masses.

Fig. 14: Dispersion relation; the energy vs. transverse momentum for $\beta = 0.45$; $g^2_{NLB} = -46$; $g_{LB} = 0$. The transverse momentum is in the $k_x - k_y$ direction.

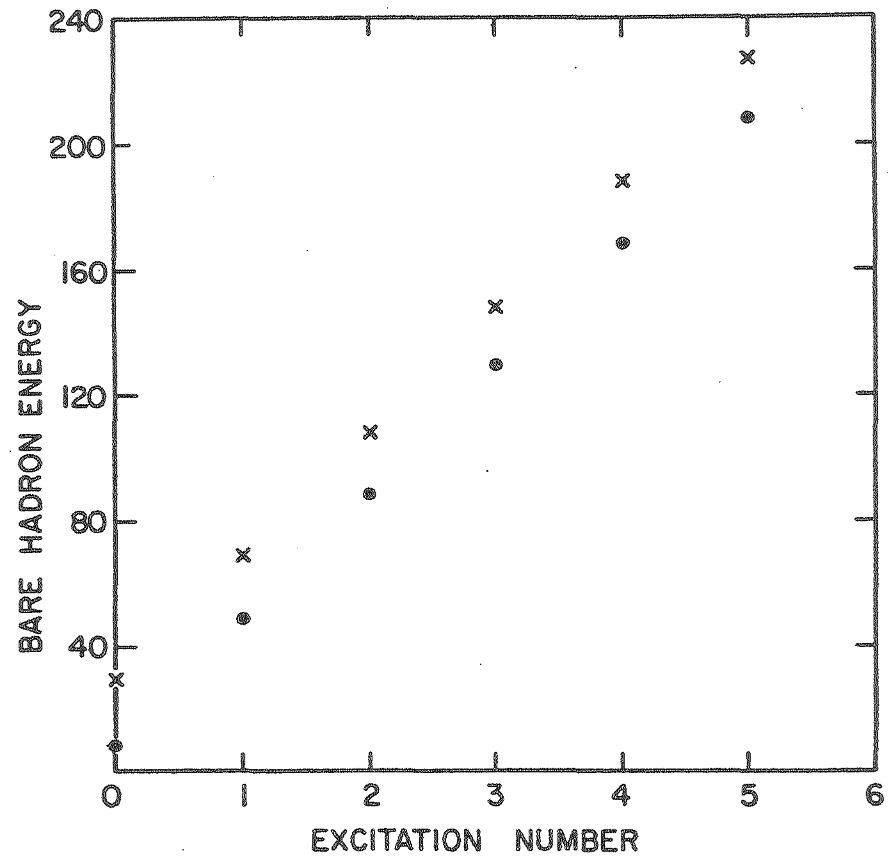


Fig. 1

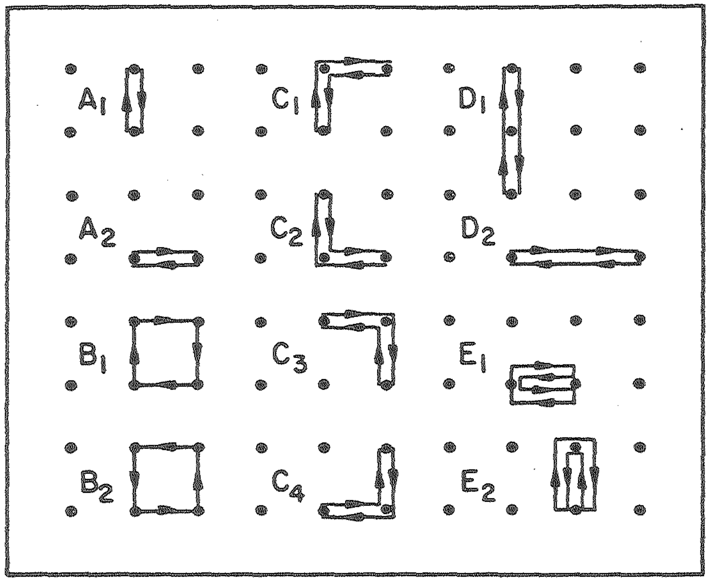


Fig. 2

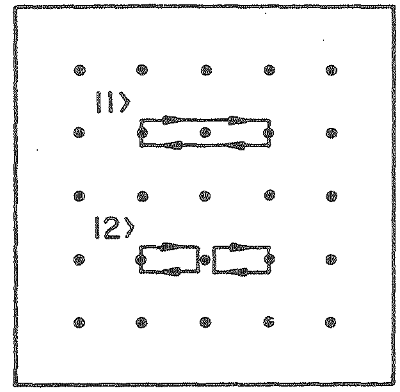
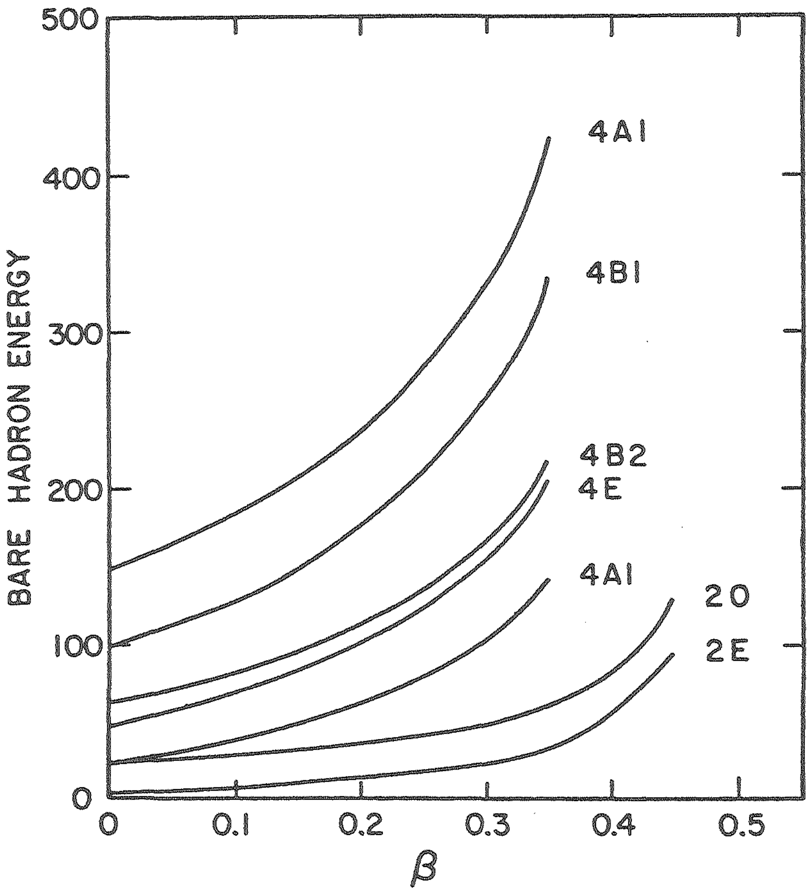
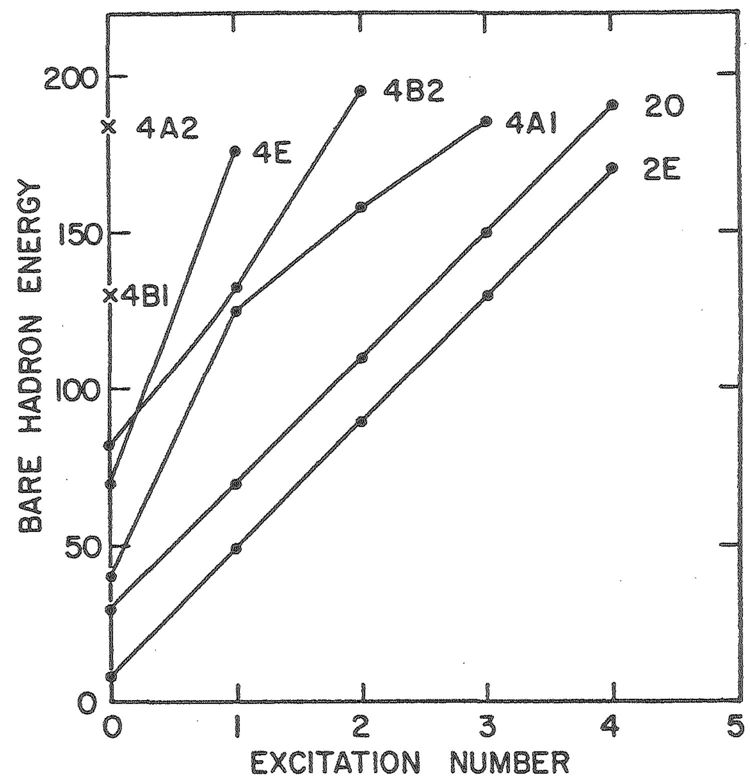


Fig. 3



BARE HADRON ENERGY
500
400
300
200
100
0
0
0.1
0.2
0.3
0.4
0.5
β
4A1
4B1
4B2
4E
4A1
2O
2E

Fig. 4



BARE HADRON ENERGY
200
150
100
50
0
0
1
2
3
4
5
EXCITATION NUMBER
× 4A2
×4B1
4E
4B2
4A1
2O
2E

Fig. 5

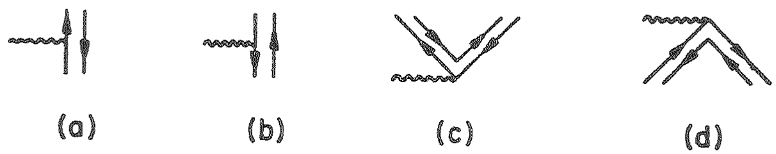


Fig. 6

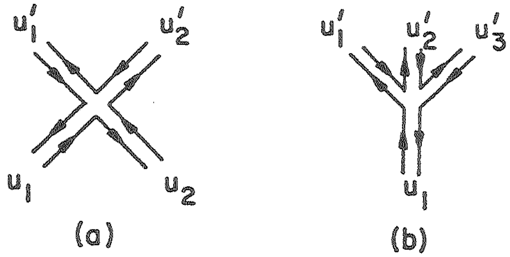


Fig. 7

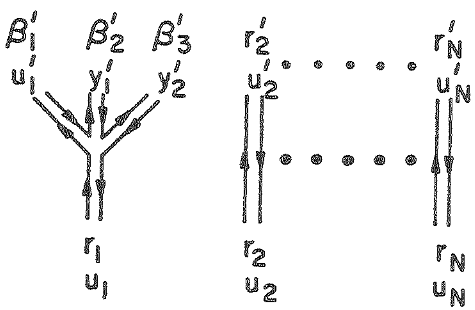


Fig. 8

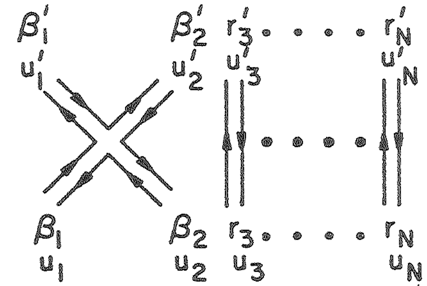


Fig. 9

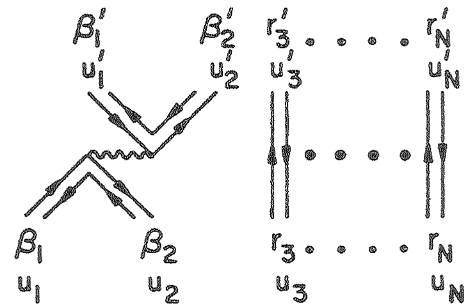


Fig. 10

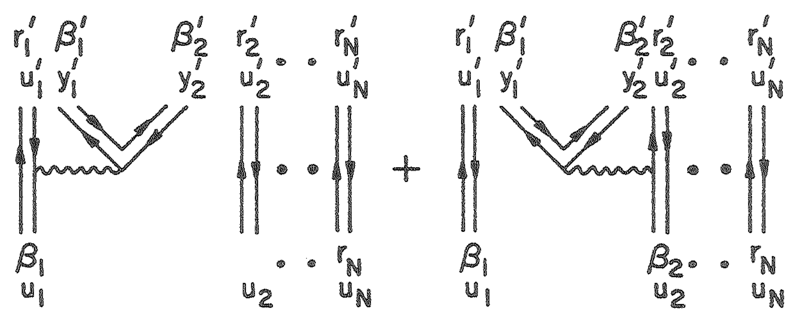


Fig. 11

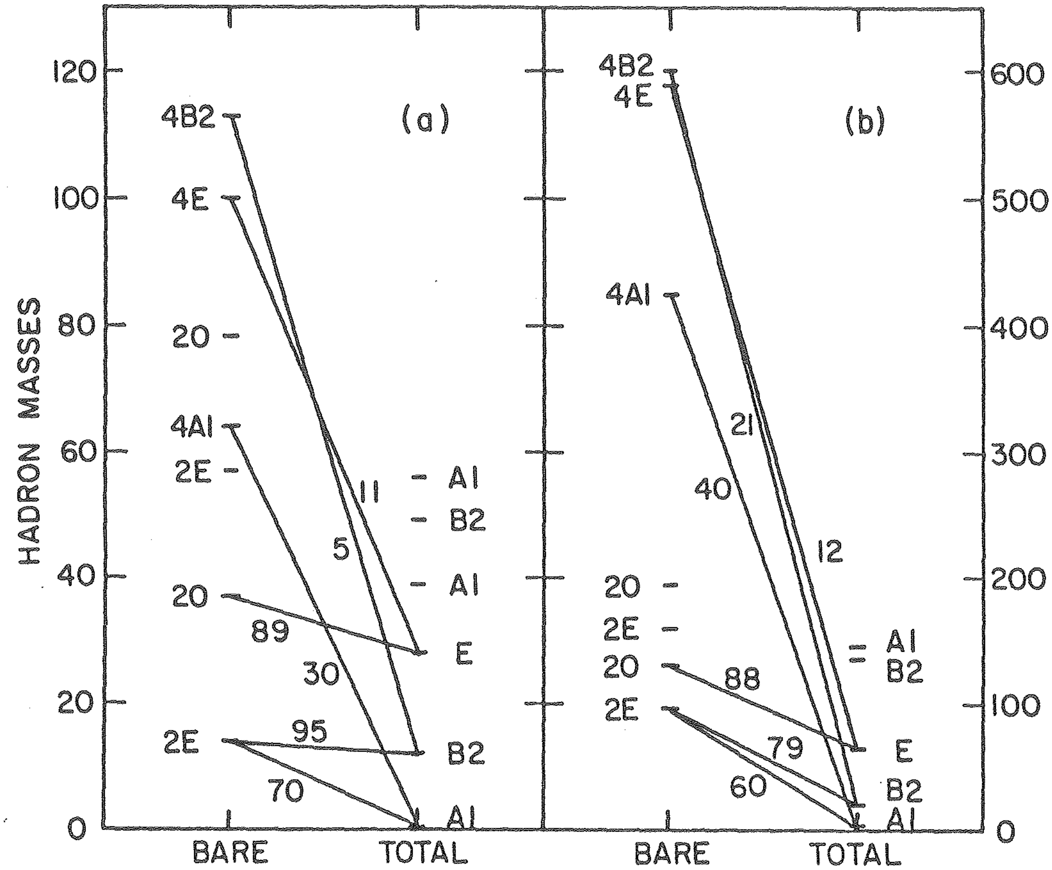

HADRON MASSES
(a)
(b)
BARE
TOTAL
4B2
4E
20
4A1
2E
A1
B2
E
89
30
95
70
11
5
21
40
12
88
79
60

Fig. 12

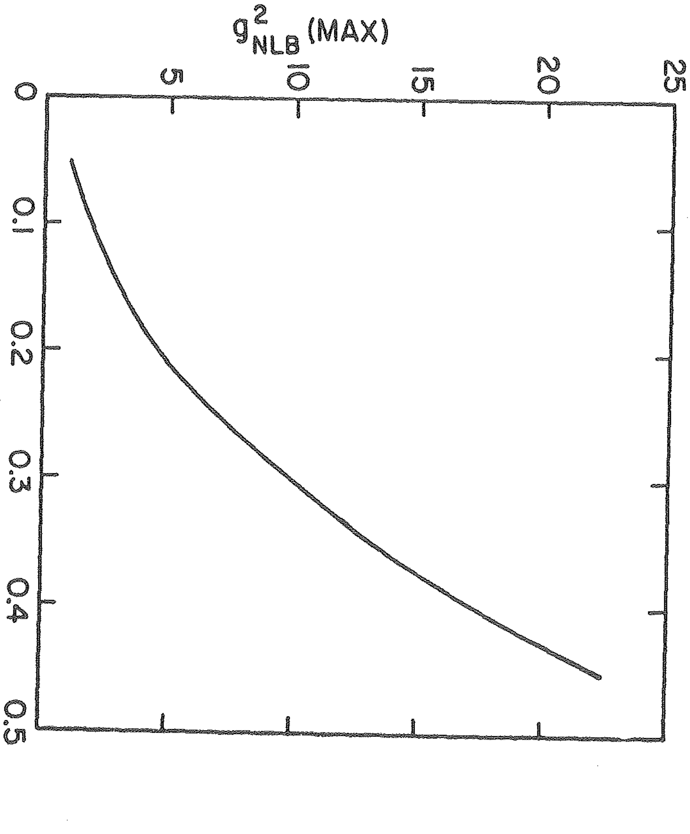

g^2_{NLB} (MAX)
β

Fig. 13

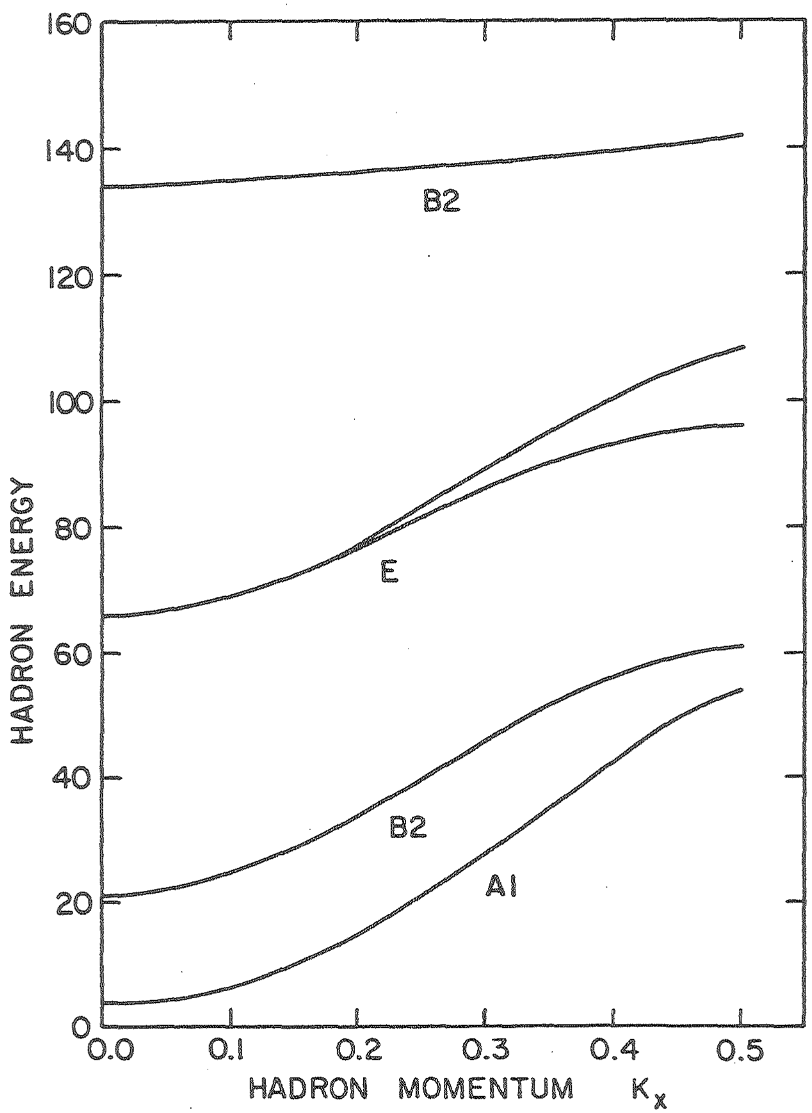

160
140
120
100
80
60
40
20
0
B2
E
B2
A1
0.0
0.1
0.2
0.3
0.4
0.5
HADRON ENERGY
HADRON MOMENTUM K_X

Fig. 14